High Order Mesh Metrics (Version 1.1)

Introduction

High order methods permit elements to have curved edges and faces. Many of the metrics commonly used with linear meshes no longer apply. One of the goals of the GMGW-3 workshop is to engage the community in a dialogue about how high-order, curved meshes can be evaluated prior to computing flow solutions. This document will provide a means for participants to share their mesh metrics with the community. If you are participating in or observing this workshop you are encouraged to 1) view this document for information regarding the mesh metrics provided by other participants and 2) add your own metric definitions. This is a Word document originally created by Steve Karman. Steve will maintain an updated version of this document. If you have edits/additions please provide them to Steve at skarman@pointwise.com.

Each metric should be defined mathematically and described in enough detail for others to implement. The purpose of the metric should be clearly stated, and the expected valid output range of values defined.

Maximum Included Angle

The maximum included angle is a metric commonly used with linear meshes to indicate elements that are near collapse. It is included in this document because the quality of the starting (linear) mesh can have a dramatic effect on the ability to create a curved mesh.

In two dimension triangles it is the maximum angle in the corners of the element, computed as the arccosine of the dot product of unit vectors emanating from the corner. For quadrilateral and arbitrary polygonal elements, the value can exceed 180 degrees in concave corners.

In three dimensions the maximum included angle is computed in corners of each face and along edges shared by faces of the element. The corner value is computed using the arccosine of the dot product of unit vectors emanating from the corner, as in the 2D case. On edges between faces of the element the unit normal vectors for each face are dotted and the angle between the normal vectors is computed using arccosine. This value is subtracted from pi to get the angle between the faces.

Jacobian

The Jacobian, in the context of mesh generation, is a mathematical term used in defining the transformation from physical space (x, y, z) to computational space (ξ, η, ζ) given by

$$J = \frac{\partial(x, y, z)}{\partial(\xi, \eta, \zeta)}$$

The mesh is invalid if the magnitude of the determinant of this matrix is zero or negative. For linear grids this represents the case where the cell volumes are less than or equal to zero using right-hand-rule notation. The finite-element formulation for the Jacobian magnitude, using shape functions N, is given by

$$J = \begin{vmatrix} \sum \frac{\partial N_i}{\partial \xi} x_i & \sum \frac{\partial N_i}{\partial \xi} y_i & \sum \frac{\partial N_i}{\partial \xi} z_i \\ \sum \frac{\partial N_i}{\partial \eta} x_i & \sum \frac{\partial N_i}{\partial \eta} y_i & \sum \frac{\partial N_i}{\partial \eta} z_i \\ \sum \frac{\partial N_i}{\partial \zeta} x_i & \sum \frac{\partial N_i}{\partial \zeta} y_i & \sum \frac{\partial N_i}{\partial \zeta} z_i \end{vmatrix}$$

For linear tetrahedral meshes the Jacobian is constant over the element. It can vary across other linear element shapes (triangular-prisms, pyramids and hexahedra). For higher order, curved elements the Jacobian must remain positive in order for the integration of the flow solution to be valid. So, for linear and higher order elements this metric must be evaluated across each element at resolutions at or higher than the flow solution integration scheme.

This metric should return all positive values and dimensional in mesh length scale units (i.e. inches/feet for CRM-HL and mm/m for JFM). The range is essentially similar to element volumes.

PW: The Pointwise mesh curving approach uses Lagrangian basis functions. The Jacobian is evaluated at 6^{th} order quadrature locations inside the element. The minimum Jacobian value is reported at completion of the mesh curving operation. In addition, the Jacobian is integrated over each element producing the volume of the element. The minimum and maximum volume is reported at completion of the mesh curving operation.

Scaled Jacobian

The scaled Jacobian is the ratio of the minimum Jacobian value inside an element divided by the maximum Jacobian value in the element. It is a measure of the variation in the Jacobian across the element. One might define it such that the denominator is always positive and non-zero. With this definition any zero or negative value indicates an invalid element. Positive values approach the ideal value of 1.

This metric should return non-dimensional values from -∞ to +1.

PW: The Pointwise mesh curving approach uses Lagrangian basis functions. The scaled Jacobian is evaluated at 6^{th} order quadrature locations inside the element. The minimum value is reported at completion of the mesh curving operation.

Shape Conformity

Linear elements have planar faces (or warped quadrilateral faces). Curved meshes improve the representation of the surface with increasing accuracy depending on the degree of the mesh. The shape conformity metric measures how well the discrete curved surface matches the underlying geometry. It is defined as the integration of the difference between the mesh surface and the geometry surface over the surface triangular or quadrilateral element. The equation below integrates the distance from a mesh point to the geometry over the surface of an element using numerical integration. The numerator results in the volume of the space between the mesh surface and the geometry. The denominator is the surface area. Combined the quantity is the average distance between the mesh and the surface.

$$SC = \frac{\iint \left|\overrightarrow{dr}\right| \, ds}{\iint ds}$$

The distance given by $\left|\overrightarrow{dr}\right|$ is the magnitude of the vector from the mesh point to the geometry, as illustrated in Figure 1.

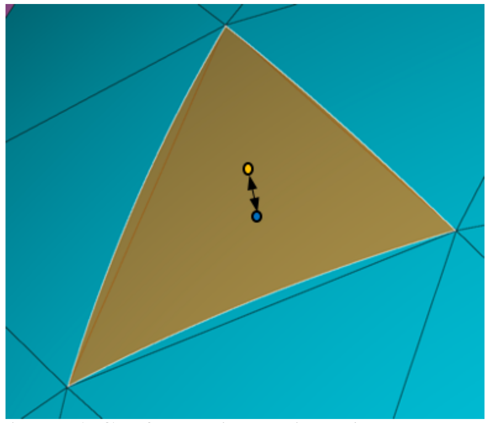

Figure 1. Surface point projected to geometry.

Numerical integration is performed using Gaussian quadrature over each surface element. For example, 6th order quadrature points are displayed in **Figure 2** for surface triangles and **Figure 3** for surface quadrilaterals.

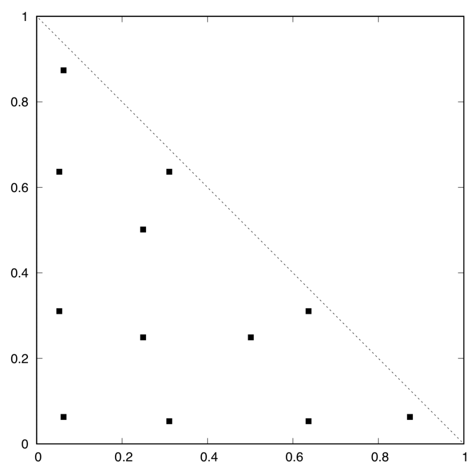

Figure 2. Sixth order Gauss points for reference triangle.

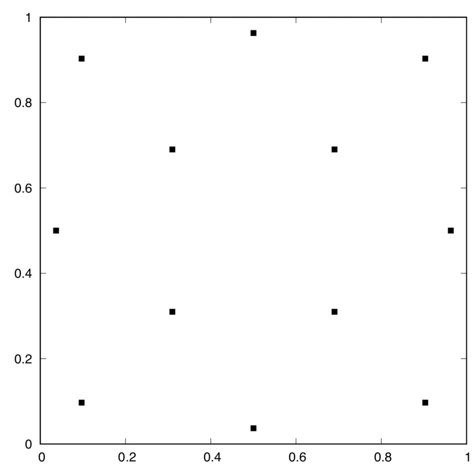

Figure 3. Sixth order Gauss points for reference quadrilateral.

The shape conformity metric is computed over each surface element.

This metric produces a dimensional quantity in the units of the mesh length scale. For flat planar surfaces all mesh orders should produce machine zero values, indicating the mesh is on the planar surface. For curved boundaries the linear mesh should exhibit the largest error. Increasing the mesh order should produce smaller error values.

PW: The Pointwise mesh curving program uses Lagrangian basis functions. The shape conformity parameter is evaluated at 6th order quadrature locations inside each surface element. Average and maximum errors are reported for each surface boundary (i.e. fuselage, wing, pylon, etc.).